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CRITICAL COMPRESSIVE STRESS FOR CURVED SHEET SUPPORTED ALONG ALL EDGES AND ELASTICALLY RESTRAINED AGAINST ROTATION ALONG THE UNLOADED EDGES

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RESTRICTED BULLETIN

CRITICAL COMPRESSIVE STRESS FOR CURVED SHEET SUPPORTED ALONG ALL EDGES AND ELASTICALLY RESTRAINED AGAINST ROTATION ALONG THE UNLOADED EDGES

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SUMMARY

A formula is given for the critical compressive stress for slightly curved sheet with equal elastic restraints against rotation along the unloaded edges. The theory of small deflections is used and the formula reduces to that given by Timoshenko for the case of simply-supported edges. For larger curvatures, a modification of Redshaw's formula to include the effect of edge restraint is suggested.

INTRODUCTION

Because the skin between stiffeners on the surface of airplanes is curved to the contour of the wing or fuselage, it is important to investigate the extent to which this curvature influences the critical stress.

The derivation and use of formulas for the critical compressive stress for flat rectangular plates with restraints against rotation along the unloaded edges were discussed in reference 1. In the present paper, the theory of small deflections is used to derive a formula for the critical compressive stress of a slightly curved sheet with restraints against rotation along the straight, unloaded edges. On the basis of this derivation, a modification of Redshaw's formula (reference 2) is suggested for cases in which the curvature is larger.

THEORY

Figure 1 shows the coordinate system and the sheet dimensions. With the assumption that the sheet is curved

to a cylindrical surface, the differential equation of the deflection surface of the sheet will be that given by Donnell as equation (10) in reference 3, which is

$$D\nabla^8 w + \frac{Et}{r^2}\frac{\partial^4 w}{\partial x^4} = -\sigma_x t\nabla^4\left(\frac{\partial^2 w}{\partial x^2}\right) \tag{1}$$

where

D flexural stiffness of sheet per unit length $\left[\frac{Et^3}{12(1-\mu^2)}\right]$

E modulus of elasticity

μ Poisson's ratio

t thickness of sheet

r radius of curvature

σ_x applied compressive stress in the x-direction

w displacement of sheet out of its original plane

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial s^2}$, $\nabla^4 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial s^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial s^2}\right)$, etc.

x longitudinal coordinate

s circumferential coordinate

The width of the sheet is $b = r\theta$, where θ is the angle subtended by the sheet at the center of curvature. The assumption is made that the critical stress σ_{cr} may be expressed in a manner similar to the critical stress of flat sheet, as

$$\sigma_{cr} = k_r\frac{\pi^2 Et^2}{12(1-\mu^2)b^2} \tag{2}$$

where k_r is a constant which depends upon the radius of curvature and the edge restraints. Substitution of σ_{cr},

as given by equation (2), for σ_x in equation (1) and division by D gives

$$\nabla^8 w + \frac{12(1-\mu^2)}{r^2t^2}\frac{\partial^4 w}{\partial x^4} = -\frac{\pi^2 k_r}{b^2}\nabla^4\left(\frac{\partial^2 w}{\partial x^2}\right) \tag{3}$$

On the assumption that the sheet is infinitely long in the x-direction, the solution may be taken in the form

$$w = f(s)\cos\frac{\pi x}{\lambda} \tag{4}$$

where

$f(s)$ function of s alone

λ half-wave length of the buckle pattern

Substitution of equation (4) in equation (3) gives as the differential equation for $f(s)$

$$\frac{d^8 f}{ds^8} - 4\left(\frac{\pi}{\lambda}\right)^2\frac{d^6 f}{ds^6} + 6\left(\frac{\pi}{\lambda}\right)^4\frac{d^4 f}{ds^4} - 4\left(\frac{\pi}{\lambda}\right)^6\frac{d^2 f}{ds^2} + \left(\frac{\pi}{\lambda}\right)^8 f$$

$$+ \frac{12(1-\mu^2)}{r^2t^2}\left(\frac{\pi}{\lambda}\right)^4 f - \frac{\pi^2 k_r}{b^2}\left[\left(\frac{\pi}{\lambda}\right)^2\frac{d^4 f}{ds^4} - 2\left(\frac{\pi}{\lambda}\right)^4\frac{d^2 f}{ds^2} + \left(\frac{\pi}{\lambda}\right)^6 f\right] = 0$$

The solution may be written

$$f(s) = c_1 e^{\alpha_r\frac{s}{b}} + c_2 e^{-\alpha_r\frac{s}{b}} + c_3 e^{i\beta_r\frac{s}{b}} + c_4 e^{-i\beta_r\frac{s}{b}}$$

$$+ c_5 e^{\alpha_r'\frac{s}{b}} + c_6 e^{-\alpha_r'\frac{s}{b}} + c_7 e^{i\beta_r'\frac{s}{b}} + c_8 e^{-i\beta_r'\frac{s}{b}} \tag{5}$$

where

$$\left.\begin{aligned}
\alpha_r &= \pi\sqrt{\frac{b}{\lambda}}\sqrt{\frac{b}{\lambda}+\sqrt{\frac{k_r}{2}+\sqrt{\frac{{k_r}^2}{4}-\frac{12(1-\mu^2)}{\pi^4}\left(\frac{b^2}{rt}\right)^2}}} \\
\beta_r &= \pi\sqrt{\frac{b}{\lambda}}\sqrt{-\frac{b}{\lambda}+\sqrt{\frac{k_r}{2}+\sqrt{\frac{{k_r}^2}{4}-\frac{12(1-\mu^2)}{\pi^4}\left(\frac{b^2}{rt}\right)^2}}} \\
\alpha_r{}' &= \pi\sqrt{\frac{b}{\lambda}}\sqrt{\frac{b}{\lambda}+\sqrt{\frac{k_r}{2}-\sqrt{\frac{{k_r}^2}{4}-\frac{12(1-\mu^2)}{\pi^4}\left(\frac{b^2}{rt}\right)^2}}} \\
\beta_r{}' &= \pi\sqrt{\frac{b}{\lambda}}\sqrt{-\frac{b}{\lambda}+\sqrt{\frac{k_r}{2}-\sqrt{\frac{{k_r}^2}{4}-\frac{12(1-\mu^2)}{\pi^4}\left(\frac{b^2}{rt}\right)^2}}}
\end{aligned}\right\} \quad (6)$$

and the c's are arbitrary constants.

As for a flat plate, there are four boundary conditions imposed on w (see reference 1). If the edges of the plate are at $s = \pm b/2$ these conditions are

$$(w)_{s=-\frac{b}{2}} = 0$$

$$(w)_{s=\frac{b}{2}} = 0$$

$$D\left(\frac{\partial^2 w}{\partial s^2} + \mu\,\frac{\partial^2 w}{\partial x^2}\right)_{s=-\frac{b}{2}} = 4S_o\left(\frac{\partial w}{\partial s}\right)_{s=-\frac{b}{2}}$$

$$D\left(\frac{\partial^2 w}{\partial s^2} + \mu\,\frac{\partial^2 w}{\partial x^2}\right)_{s=\frac{b}{2}} = -4S_o\left(\frac{\partial w}{\partial s}\right)_{s=\frac{b}{2}}$$

where S_0 is the stiffness per unit length of the elastic restraining medium or structural element at the edges of the sheet; that is, S_0 is the ratio of moment per unit length at any point along the length of the medium or structural element to the rotation in quarter-radians at that point when the moment is distributed sinusoidally. The requirement of a sinusoidally distributed moment is satisfied when an infinitely long plate buckles under longitudinal compression.

Of the eight solutions summed in equation (5), only four are required to satisfy the foregoing conditions. The question of which solutions to use is decided by the requirement that the deflection surface reduce to that for a flat sheet when the radius of curvature is made to approach infinity. Putting $r = \infty$ in equations (6) gives

$$\left.\begin{aligned}
\alpha_\infty &= \pi\sqrt{\frac{b}{\lambda}}\sqrt{\frac{b}{\lambda} + \sqrt{k_\infty}} \\
\beta_\infty &= \pi\sqrt{\frac{b}{\lambda}}\sqrt{-\frac{b}{\lambda} + \sqrt{k_\infty}} \\
\alpha_\infty' &= \pi\frac{b}{\lambda} \\
\beta_\infty' &= i\pi\frac{b}{\lambda}
\end{aligned}\right\} \qquad (7)$$

where k_∞ is the value of k_r when $r = \infty$. The quantities α_∞ and β_∞ are seen to be identical with the α and β of reference 1; whereas α_∞' and β_∞' have no counterpart for flat plates. The deflection surface for the curved plate may thus be written

$$w = \left(c_1 e^{\alpha_r \frac{s}{b}} + c_2 e^{-\alpha_r \frac{s}{b}} + c_3 e^{i\beta_r \frac{s}{b}} + c_4 e^{-i\beta_r \frac{s}{b}}\right)\cos\frac{\pi x}{\lambda}$$

In addition to satisfying boundary conditions involving w, the effect of curvature requires that conditions involv-

ing the displacements u and v in the longitudinal and circumferential directions also be satisfied. Disregarding the four solutions in α_r' and β_r' may be interpreted physically as neglecting this effect of curvature. This neglect is permissible for plates of small curvature. (See reference 3.)

The theory of reference 1 shows that, for a flat plate with an elastic restraint of stiffness S_0 along each unloaded edge, the critical buckling stress is to be found from the expression

$$\frac{\alpha_\infty^2 + \beta_\infty^2}{\alpha_\infty \tanh \frac{\alpha_\infty}{2} + \beta_\infty \tan \frac{\beta_\infty}{2}} = -\epsilon \tag{8}$$

where ϵ is the restraint coefficient, given as

$$\epsilon = \frac{4S_0 b}{D}$$

The deflection surface and boundary conditions used here are formally the same as for flat sheet; therefore, for a curved sheet with the same restraining members

$$\frac{\alpha_r^2 + \beta_r^2}{\alpha_r \tanh \frac{\alpha_r}{2} + \beta_r \tan \frac{\beta_r}{2}} = -\epsilon \tag{9}$$

It may be seen from a comparison of the definitions of α_∞, β_∞ and α_r, β_r as given in equations (7) and (6) that, if equation (8) is satisfied when k_∞ has some particular value, equation (9) will be satisfied when the expression

$$\frac{k_r}{2} + \sqrt{\frac{k_r^2}{4} - \frac{12(1-\mu^2)}{\pi^4}\left(\frac{b^2}{rt}\right)^2}$$

has this same value. A relation between the initial buckling stresses for curved and for flat sheet can therefore be obtained by setting

$$\frac{k_r}{2} + \sqrt{\frac{k_r^2}{4} - \frac{12(1 - \mu^2)}{\pi^4}\left(\frac{b^2}{rt}\right)^2} = k_\infty$$

or

$$k_r = k_\infty + \frac{12(1 - \mu^2)}{\pi^4 k_\infty}\left(\frac{b^2}{rt}\right)^2$$

Good for $\frac{b^2}{Rt}\sqrt{1-\mu^2} < 10$, for s.s. edges.

By substitution of this value of k_r in equation (2) the critical stress is found to be

$$\sigma_{cr} = \left[k_\infty + \frac{12(1 - \mu^2)}{\pi^4 k_\infty}\left(\frac{b^2}{rt}\right)^2\right]\frac{\pi^2 E t^2}{12(1 - \mu^2) b^2}$$

$$= \frac{k_\infty \pi^2 E t^2}{12(1 - \mu^2) b^2} + \frac{\theta^2 E}{\pi^2 k_\infty} \qquad (10)$$

where the first term is the critical compressive stress of the sheet when flat and k_∞ is the coefficient k for the flat sheet, as determined by the methods of reference 1. For the case of simple support along the edges, $k_\infty = 4$ and equation (10) reduces to a formula given by Timoshenko (reference 4, p. 470, equation (276)).

Redshaw has proposed the following approximate formula, derived by an energy method without limitations as to curvature, for simply supported edges (reference 2, equation (31)):

$$\sigma_{cr} = \frac{4\pi^2 E t^2}{12(1-\mu^2)b^2} \; \frac{1 + \sqrt{1 + \frac{12(1-\mu^2)}{\pi^4}\left(\frac{b^2}{rt}\right)^2}}{2} \tag{11}$$

For curvatures so small that the radical may be expanded in series, the Redshaw formula reduces to

$$\sigma_{cr} = \frac{4\pi^2 E t^2}{12(1-\mu^2)b^2} + \frac{\theta^2 E}{\pi^2} \tag{12}$$

which is of the same form as equation (10) but does not include the coefficient k_∞ in the term that takes account of the curvature.

The formal relation between equations (11) and (12) suggests that an equation similar to equation (11) exists that will yield equation (10) when the curvature is small. Such an equation is

$$\sigma_{cr} = \frac{k_\infty \pi^2 E t^2}{12(1-\mu^2)b^2} \; \frac{1 + \sqrt{1 + \frac{48(1-\mu^2)}{\pi^4}\left(\frac{b^2}{k_\infty r t}\right)^2}}{2} \tag{13}$$

Equation (13), although not actually derived, appears to be a reasonable generalization of the formula of Redshaw and yields the following results in special cases:

Flat plate; $\frac{b^2}{rt} = 0$; $\qquad \sigma_{cr} = \frac{k_\infty \pi^2 E t^2}{12(1-\mu^2)b^2}$

Slight curvature; $\frac{b^2}{rt}$ small; $\sigma_{cr} = \frac{k_\infty \pi^2 E t^2}{12(1-\mu^2)b^2} + \frac{\theta^2 E}{\pi^2 k_\infty}$

Large curvature; $\frac{b^2}{rt}$ large; $\sigma_{cr} = \frac{E\frac{t}{r}}{\sqrt{12(1-\mu^2)}} = 0.303E\frac{t}{r}$ (for $\mu = 0.3$)

The value of σ_{cr} for large curvature is half the classical value for a complete cylinder and is in better agreement with cylinder tests than the classical value. Equation (13) might, therefore, be expected to hold reasonably well for all curvatures and for any degree of restraint at the edges of the sheet. The tests that have been made (reference 5) indicate, however, that the effect of curvature cannot always be relied upon to follow consistently the gradual increase in critical stress with increase in curvature represented by equation (13).

CONCLUSION

The critical compressive stress for an infinitely long slightly curved sheet with equal elastic restraints against rotation along the unloaded edges, as obtained from the differential-equation solution, is given by the equation

$$\sigma_{cr} = \frac{k_\infty \pi^2 E t^2}{12(1 - \mu^2) b^2} + \frac{\theta^2 E}{\pi^2 k_\infty}$$

where the first term gives the critical compressive stress for the sheet when flat and

k_∞ coefficient k determined for flat sheet (by method of reference 2)

θ angle subtended by sheet at center of curvature

E modulus of elasticity

t thickness of sheet

μ Poisson's ratio

b width of sheet

For larger curvatures, the formula

$$\sigma_{cr} = \frac{k_\infty \pi^2 E t^2}{12(1 - \mu^2) b^2} \; \frac{1 + \sqrt{1 + \frac{48(1 - \mu^2)}{\pi^4}\left(\frac{b^2}{k_\infty r t}\right)^2}}{2}$$

is suggested, where r is the radius of curvature.

The tests that have been made indicate, however, that the effect of curvature cannot always be relied upon to follow consistently the gradual increase in critical stress with increase in curvature represented by these equations.

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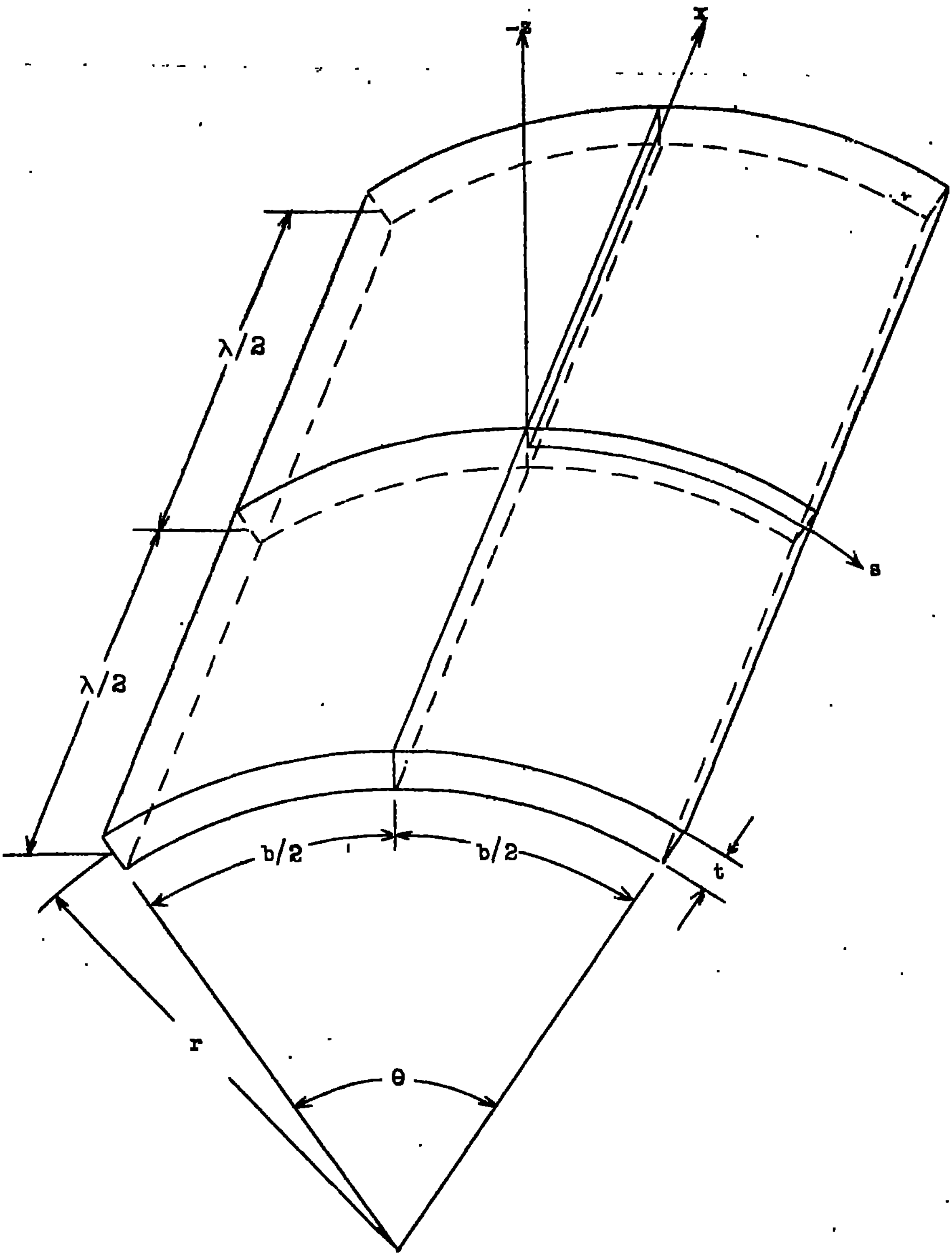

Figure 1.- Coordinate system for curved sheet.